SECTION – IV

DIFFERENTIATION AND INTEGRATION

The "fundamental theorem of the integral calculus" is that differentiation and integration are inverse processes. This general principle may be interpreted in two different ways.

If f(x) is integrable, the function

$F(x) = \int_a^x f(t)dt$

is called the indefinite integral of f(x); and the principle asserts that(i) $\acute{F}(x) = f(x)$ (ii)

On the other hand, if F(x) is a given function, and f(x) is defined by (ii), the principle asserts that

$\int_a^x f(t)dt = F(x) - F(a)$ (iii)

The main object of this chapter is to consider in what sense these theorems are true.

From the theory of Riemann integration (ii) follows from (i) if x is a point of continuity of f. For we can choose h_0 so small that $|f(t) - f(x)| < \varepsilon$ for $|t-x| \le h_0$; and the

$\frac{F(x+h)-F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h}\{f(t) - f(x)\}dt \le \varepsilon$ ($|h| < h_0$) by the mean-value theorem.

This proves (ii).

We shall show that more generally this relation holds almost everywhere. Thus differentiation is the inverse of Lebesgue integration.

The problem of deducing (iii) from (ii) is more difficult and even using Lebesgue integral it is true only for a certain class of functions. We require in the first place that $\acute{F}(x)$should exist at any rate almost everywhere and as we shall see this is not necessarily so. Secondly, if $\acute{F}(x)$ exists we require that it should be integrable.

Differentiation of Monotone Functions

4.1. Definition. Let C be a collection of intervals. Then we say that C covers a set E in the sense of Vitali, if for each $\varepsilon > 0$ and x in E there is an interval I ∈ C such that x∈I and $l(I) < \varepsilon$.

Now we prove the following lemma which will be utilized in proving a result concerning the differentiation of monotone functions.

4.2. Lemma. (Vitali). Let E be a set of finite outer measure and C a collection of intervals which cover E in the sense of Vitali. Then given $\varepsilon > 0$ there is a finite disjoint collection $\{I_1,\ldots, I_n\}$ of intervals in C such that

$$m^*[E - \cup_{n=1}^{N} I_n] < \varepsilon.$$

Proof. It suffices to prove the lemma in the case that each interval in C is closed, for otherwise we replace each interval by its closure and observe that the set of endpoints of $I_1, I_2, \ldots, I_N$ has measure zero.

Let O be an open set of finite measure containing E. Since C is a Vitali covering of E, we may suppose without loss of generality that each I of C is contained in O. We choose a sequence $< I_n >$ of disjoint intervals of C by induction as follows :

Let I_1 be any interval in C and suppose $I_1, \ldots, I_n$ have already been chosen. Let k_n be the supremum of the lengths of the intervals of C which do not meet any of the intervals $I_1, \ldots, I_n$.

Since each I is contained in O, we have $k_n \leq m\,O < \infty$. Unless, $E \subset \cup_{i=1}^{n} I_I$ we can find I_{n+1} in C with $l(I_{n+1}) > \frac{1}{2} k_n$ and I_{n+1} is disjoint from $I_1, I_2, \ldots, I_n$. Thus we have a sequence $< I_n >$ of disjoint intervals of C, and since $\cup I_n \subset O$, we have $\sum l(I_n) \leq m\,O < \infty$.

Hence we can find an integer N such that $\sum_{N+1}^{\infty} l(I_n) < \frac{\varepsilon}{5}$

Let $R = E - \cup_{n=1}^{N} I_n$.

It remains to prove that $m^*R <$.

Let x be an arbitrary point of R. Since $\cup_{n=1}^{N} I_n$ is a closed set not containing x, we can find an

interval I in C which contains x and whose length is so small that I does not meet any of the intervals $I_1, I_2, \ldots, I_N$. If now $I \cap I_i = \emptyset$ for $i \leq N$, we must have $l(I) \leq k_N < 2l(I_{N+1})$. Since $\lim l(I_n) = 0$, the interval I must meet at least one of the intervals I_n. Let n be the smallest integer such that I meets I_n.

We have $n > N$, and $l(I) \leq k_N \leq 2l(I_{N+1})$. Since x is in I, and I has a point in common with I_n, it follows that the distance from x to the midpoint of I_n is at most $l(I) + \frac{1}{2} l(I_N) \leq \frac{5}{2} l(I_{N+1})$.

Let J_m denote the interval which has the same midpoint as I_m and five times the length of I_m. Then we have $x \in J_m$. This proves $R \subset \cup_{N+1}^{\infty} J_n$

Hence $m^* R \leq \sum_{N+1}^{\infty} l(J_n) = 5 \sum_{N+1}^{\infty} l(J_n) < \varepsilon.$

The Four Derivatives of a Function

Whether the differential coefficients

$$\acute{f}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exist or not, the four expressions

$$D^{+}f(x) = \overline{\lim_{h \to 0+}} \frac{f(x+h) - f(x)}{h}$$

$$D^{-}f(x) = \overline{\lim_{h \to 0+}} \frac{f(x) - f(x-h)}{h}$$

$$D_{+}f(x) = \underline{\lim}_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$

$$D_{-}f(x) = \underline{\lim}_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$

always exist. These derivatives are known as Dini Derivatives of the function f.

$D^+ f(x)$ and $D_+ f(x)$ are called upper and lower derivatives on the right and $D^- f(x)$ and $D_- f(x)$ are **called upper and lower derivatives on the left.** Clearly we have $D^+ f(x) \geq D_+ f(x)$ and $D^- f(x) \geq D_- f(x)$. If $D^+ f(x) = D_+ f(x)$, the function f is said to have a **right hand derivative** and if $D^- f(x) = D_- f(x)$, the function is said to have a **left hand derivative.**

If $D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \neq \mp\infty$ we say that f is differentiable at x and define f'(x) to be the common value of the derivatives at x.

4.3.Theorem. Every non-decreasing function f defined on the interval [a, b] is differentiable almost everywhere in [a, b]. The derivative f' is measurable and

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Proof. We shall show first that the points *x* of the open interval (a, b) at which **not** all of the four Dini-derivatives of *f* are equal form a subset of measure zero. It suffices to show that the following four subsets of (a, b) are of measure zero:

$A = \{x \in (a, b) \mid D_- f(x) < D^+ f(x)\}$,

$B = \{x \in (a, b) \mid D_+ f(x) < D^- f(x)\}$,

$C = \{x \in (a, b) \mid D_- f(x) < D^- f(x)\}$,

$D = \{x \in (a, b) \mid D_+ f(x) < D^+ f(x)\}$. To prove $m^* A = 0$, consider the subsets

$A_{u,v} = \{x \in (a, b) \mid D_- f(x) < u < v < D^+ f(x)\}$

of A for all rational numbers u and v satisfying $u < v$. Since A is the union of this countable family $\{A_{u,v}\}$, it is sufficient to prove $m^*(A_{u,v}) = 0$ for all pairs u, v with $u < v$.

For this purpose, denote $\alpha = m^*(A_{u,v})$ and let ε be any positive real number. Choose an open set $U \supset A_{u,v}$ with $m^* U < \alpha + \varepsilon$. Set x be any point of $A_{u,v}$. Since $D_- f(x) < u$, there are arbitrary small closed intervals of the form [x-h, x] contained in U such that

$f(x) - f(x-h) < uh$.

Do this for all $x \in A_{u,v}$ and obtain a Vitali cover C of $A_{u,v}$. Then by Vitali covering theorem there is a finite subcollection $\{J_1, J_2, \ldots, J_n\}$ of disjoint intervals in C such that

$m^*(A_{u,v} - \cup_{i=1}^{n} J_i) < \varepsilon$

Summing over these n intervals, we obtain

$$\begin{aligned}\sum_{i=1}^{n} f x_i - f(x_i h_i) &< u \sum_{i=1}^{n} h_i \\ &< u\, m^* U \\ &< u(\alpha + \varepsilon)\end{aligned}$$

Suppose that the interiors of the intervals $J_1, J_2, \ldots, J_n$ cover a subset F of $A_{u,v}$. Now since $D^+ f(y) > v$, there are arbitrarily small closed intervals of the form [y, y+k] contained in some of the intervals J_i (i = 1, 2,…, n) such that f(y+k) − f(y) > vk

Do this for all y ∈ F and obtain a Vitali cover D of F. Then again by Vitali covering lemma we can select a finite subcollection [K_1, K_2, …, K_m] of disjoint intervals in D such that

$m^*[F - \cup_{i=1}^{m} K_i] < \varepsilon$

Since m*F > $\alpha - \varepsilon$, it follows that the measure of the subset H of F which is covered by the intervals is greater than $\alpha - 2\varepsilon$. Summing over these intervals and keeping in mind that each K_i is contained in a J_n, we have

$$\sum_{i=1}^{n}\{f x_i - f(x_i - h_i)\} \geq \sum_{i=1}^{m} [f(y_i + k_i) - f y_i]$$
$$> v \sum_{i=1}^{n} k_i$$
$$> v(\alpha - 2\varepsilon)$$

So that

$$v(\alpha - 2\varepsilon) < u(\alpha - \varepsilon)$$

Since this is true for every $\varepsilon > 0$, we must have v $\alpha <$ u α. Since u < v, this implies that $\alpha = 0$. Hence m*A = 0. Similarly, we can prove that m*B = 0, m*C = 0 and m*D = 0. This shows that

$$g(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is defined almost everywhere and that f is differentiable whenever g is finite.

If we put

$$g_n(x) = n[f(x+\tfrac{1}{n}) - f(x)] \qquad \text{for } x \in [a,b] ,$$

where we re-define f(x) = f(b) for x ≥ b. Then $g_n(x) \to g(x)$ for almost all x and so g is measurable since every g_n is measurable. Since f is non-decreasing, we have $g_n \geq 0$. Hence, by

Fatou's Lemma

$$\int_a^b g \leq \underline{lim} \int_a^b g_n = \underline{lim}\ n \int_a^b [f\left(x + \tfrac{1}{n}\right) - f(x)]dx$$

$$= \underline{lim}\ n \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx$$

$$= \underline{lim}\ n \int_a^b [f(x) + \int_b^{b+\frac{1}{n}} f(x)dx - \int_a^{a+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx$$

$$= \underline{lim}\ n \int_a^b [f\left(x + \tfrac{1}{n}\right) - f(x)]dx$$

$$\leq f(b)-f(a)$$

(Use of f(x) = f(b) for x ≥ b for first interval and f non-decreasing in the 2nd integral).

This shows that g is integrable and hence finite almost everywhere. Thus f is differentiable almost everywhere and g(x) = $\acute{f}$(x) almost everywhere. This proves the theorem.

Functions of Bounded Variation

Let f be a real-valued function defined on the interval [a,b] and let $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be any partition of [a,b].

By the variation of f over the partition $P = \{x_0, x_1, \ldots, x_n\}$ of [a,b], we mean the real number $V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$

and then

$$V_a^b(f) = \sup \{V(f,P) \text{ for all possible partitions P of } [a,b]\}$$
$$= sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

is called the total variation of f over the interval [a,b]. If $V_a^b(f) < \infty$ then we say that f is a function of bounded variation and we write f ∈ BV.

4.4. Lemma. Every non-decreasing function f defined on the interval [a,b] is of bounded variation with total variation

$$V_a^b(f) = f(b) - f(a).$$

Prof. For every partition $P = [x_0, x_1, \ldots, x_n\}$ of [a,b], we have

$$V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$
$$= f(b) - f(a)$$

This implies the lemma.

4.5.Theorem. (Jordan Decomposition Theorem). A function f: [a,b] → **R** is of bounded variation if and only if it is the difference of two non-decreasing functions.

Proof. Let f = g-h on [a,b] with g and h increasing. Then for any, subdivision we have

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{n} |h(x_i) - h(x_{i-1})|$$
$$= g(b) - g(a) + h(b) - h(a)$$

Hence,

$$V_a^b(f) \leq g(b) - g(a) + h(b) - h(a),$$

which proves that f is of bounded variations.

On the other hand, let f be of bounded variation. Define two functions g, h : [a, b] →R by taking

$$g(x) = V_a^x(f), \quad h(x) = V_a^x(f) - f(x) \quad \text{for every } x \in [a, b]. \text{ Then } f(x) = g(x) - h(x).$$

The function g is clearly non-decreasing. On the other hand, for any two real numbers x and y in [a, b] with $x \leq y$, we have

$$h(y) - h(x) = [V_a^x(f) - f(y)] - [V_a^x(f) - f(x)]$$

$= V_x^{\,y}(f) - [f(y) - f(x)]$

$\geq V_x^{\,y}(f) - V_x^{\,y}(f) = 0$

Hence h is also non-decreasing. This completes the proof of the theorem.

4.6. Examples. (1) If f is monotonic on [a,b], then f is of bounded variation on [a, b] and V(f) = |f(b)-f(a)| , where V(f) is the total variation.

(2) If $\acute{f}(x)$ exists and is bounded on [a, b], then f is of bounded variation. For if $|\acute{f}(x)| \leq M$ we have

$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} M|x_i - x_{i-1}| = M(b-a)$

no matter which partition we choose.

(3) *f* may be continuous without being of bounded variation. Consider

$$f(x) = \begin{cases} x \sin\frac{\pi}{x} & (0 < x \leq 2) \\ 0 & (x = 0) \end{cases}$$

Let us choose the partition which consists of the points

$$0, \frac{2}{2^{n-1}}, \frac{2}{2^{n-3}}, \frac{2}{2^{n-5}}, \ldots\ldots \frac{2}{5}, \frac{2}{3}, 2$$

Then the sum in the total variation is

$$\left(2 + \frac{2}{3}\right) + \left(\frac{2}{3} + \frac{2}{5}\right) + \ldots.. + \left(\frac{2}{2^{n-3}} + \frac{2}{2^{n-1}}\right) + \frac{2}{2^{n-1}} > \frac{1}{2} + \frac{1}{3} + \ldots.. + \frac{1}{n}$$

and this can be made arbitrarily large by taking n large enough, since $\sum \frac{1}{n}$ diverges.

(4). Since $|f(x) - f(a)| \leq V(f)$ for every x on [a,b] it is clear that every function of bounded variation is bounded.

The Differentiation of an Integral

Let f be integrable over [a,b] and let

$F(x) = \int_a^x f(t)dt$

If f is positive, h > 0 , then we see that

$F(x+h) - F(x) = \int_x^{x+h} f(t)dt$

Hence, integral of a positive function is non-decreasing.

We shall show first that F is a function of bounded variation. Then, being function of bounded variation, it will have a finite differential coefficient F' almost everywhere. Our object is to prove that $\acute{F}(x) = f(x)$ almost everywhere in [a,b]. We prove the following lemma :

4.7. Lemma. If f is integrable on [a,b], then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is a continuous function of bounded variation on [a,b].

Proof. We first prove continuity of F. Let x_0 be an arbitrary point of [a,b]. Then

$|F(x) - F(x_0)| = |\int_{x_0}^{x} f(t)dt \ |$

$$\leq \int_{x_0}^{x} |f(t)|dt$$

Now the integrability of *f* implies integrability of |f| over [a,b]. Therefore, given $\varepsilon > 0$ there is a $\delta > 0$ such that for every measurable set A $\subset$[a,b] with measure less than δ, we have $\int_A |f| < \varepsilon$.Hence

$|F(x)-F(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta_1$

and so f is continuous.

To show that F is of bounded variation, let a = $x_0 < x_1 < \ldots < x_n$ = b be any partition of [a,b]. Then

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} | \int_a^{x_i} f(t)dt - \int_a^{x_{i-1}} f(t)dt \ |$$

$$= \sum_{i=1}^{n} | \int_{x_{i-1}}^{x_i} f(t)dt \ |$$

$$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|dt$$

$$= \int_a^b |f(t)|dt$$

Thus $V_a^b(f) \leq \int_a^b |f(t)|dt < \infty$

Hence F is of bounded variation.

4.8. Lemma. If f is integrable on [a, b] and

$$\int_a^x f(t)dt = 0$$

for all x ∈ [a,b], then f = 0 almost everywhere in [a,b].

Proof. Suppose f > 0 on a set E of positive measure. Then there is a closed set F ⊂ E with m F > 0. Let O be the open set such that

O = (a, b) – F

Then either $\int_a^b f \neq 0$ or else

$$0 = \int_a^b f = \int_A f + \int_O f$$

$$= \int_F f + \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(t)dt \qquad (1)$$

because O is the union of a countable collection $\{(a_n, b_n)\}$ of open intervals.

But, for each n,

$$\int_{a_n}^{b_n} f(t)dt = \int_a^{b_n} f(t)dt - \int_a^{a_n} f(t)dt$$

= $F(b_n) - F(a_n)$ = 0 (by hypothesis) Therfore, from (1), we have

$$\int_F f = 0$$

But since f > 0 on F and mF > 0, we have $\int_F f > 0$.

We thus arrive at a contradiction. Hence f = 0 almost everywhere.

4.9. Lemma. If f is bounded and measurable on [a, b] and

$$F(x) = \int_F^x f(t)dt + F(a),$$

then F'(x) = f(x) for almost all x in [a,b]. $f_n(x) = \frac{1}{h}[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt]$

Proof. We know that an integral is of bounded variation over [a,b] and so F'(x) exists for almost all x in [a,b]. Let |f| ≤ K. We set

$$f_n(x) = \frac{F(x+h)-F(x)}{h}$$

with $h = \frac{1}{n}$. Then we have

$$f_n(x) = \frac{1}{h}[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt]$$

$$= \frac{1}{h}[\int_x^{x+h} f(t)dt$$

implies $|f_n(x)| = |\frac{1}{h}[\int_x^{x+h} f(t)dt| \le \frac{1}{h}\int_x^{x+h} |f(t)|dt$

$$\le \frac{1}{h}\int_x^{x+h} K\,dt$$

$= \frac{K}{h} . h = K$

Moreover,

$$f_n(x) \to F'(x)$$

Hence by the theorem of bounded convergence, we have

$$\int_a^c F'(x)dx = \lim \int_a^c f_n(x)dx = \lim_{h\to 0} \frac{1}{h}\int_a^c [F(x+h) - F(x)]dx$$

$$= \lim_{h\to 0} [\frac{1}{h}\int_{a+h}^{c+h} F(x)dx - \frac{1}{h}\int_a^c F(x)\,dx]$$

$$= \lim_{h\to 0} [\frac{1}{h}\int_c^{c+h} F(x)dx - \frac{1}{h}\int_a^{a+h} F(x)\,dx]$$

$$= F(c) - F(a)$$

$$= \int_a^c f(x)dx$$

Hence,

$$\int_a^c [F'(x) - f(x)]dx = 0$$

For all c∈[a,b], and so

$$F'(x) = f(x) \text{ a.e.}$$

By using pervious lemma.

Now we extend the above lemma to unbounded functions.

4.10. Theorem. Let f be an integrable function on [a,b] and suppose that

$$F(x) = F(a) + \int_a^x f(x)dt$$

Then F'(x) = f(x) for almost all in x in [a,b].

Proof. Without loss of generality we may assume that $f \geq 0$ (or we may write "From the definition of integral it is sufficient to prove the theorem when $f \geq 0$).

Let f_n be defined by $f_n(x) = f(x)$ if $f(x) \leq n$ and $f_n(x) = n$ if $f(x) > n$. Then $f - f_n \geq 0$ and so

$$G_n(x) = \int_a^x (f - f_n)$$

is an increasing function of x, which must have a derivative almost everywhere and this derivative will be non-negative. Also by the above lemma, since f_n is bounded (by n), we have

$$\frac{d}{dx}\left(\int_a^x f_n\right) = f_n(x) \text{ a.e.}$$

Therefore,

$$F'(x) = \frac{d}{dx}\left(\int_a^x f\right) = \frac{d}{dx}\left(G_n + \int_a^x f_n\right)$$

$$= \frac{d}{dx}\left(G_n + \frac{d}{dx}\left(\int_a^x f_n\right) \geq f_n(x) \quad \text{a.e.} \qquad \text{(using (i))}$$

Since n is arbitrary, making $n \to \infty$ we see that

$F'(x) \geq f(x)$ a.e.

Consequently,

$$\int_a^b F'(x)dx \geq \int_a^b f(x)dx = F(b) - F(a) \qquad \text{(using the hypothesis of the theorem)}$$

Also since F(x) is an increasing real valued function on the interval [a,b], we have

$$\int_a^b F'(x)dx \leq F(b) - F(A) = \int_a^b f(x)dx$$

Hence

$$\int_a^b F'(x)dx = F(b) - F(A) = \int_a^b f(x)dx$$

implies $\int_a^b [F'(x) - f(x)]dx = 0$

Since $F'(x) - f(x) \geq 0$, this implies that $F'(x) - f(x) = 0$ a.e. and so $F'(x) = f(x)$ a.e.

Absolute Continuity

4.11. Definition. A real-valued function f defined on [a,b] is said to be **absolutely continuous** on [a,b] if, given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - F(x_i)| < \varepsilon$$

for every finite collection $\{(x_i, x_i')\}$ of non-overlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta$$

An absolutely continuous function is continuous, since we can take the above sum to consist of one term only. Moreover, if

$$F(x) = \int_a^x f(t)dt$$

Then

$$\sum_{i=1}^{n} |f(x_i') - F(x_i)| = \sum_{i=1}^{n} \left| \int_a^{x_i'} f(t)dt - \int_a^{x_i} f(t)dt \right|$$

$$= \sum_{i=1}^{n} \left| \int_{x_i}^{x_i'} f(t)dt \right|$$

$$\leq \sum_{i=1}^{n} \int_{x_i}^{x_i'} |f(t)|dt = \int_E |f(t)|dt ,$$

where E is the set of intervals (x,x_i') $\leq$ $\to 0$ as $\sum_{i=1}^{n} |x_i' - x_i| \to 0$

The last step being the consequence of the result.

"Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that for every measurable set $E \subset [a, b]$ with

m E < δ, we have $\int_A | f | < \varepsilon$".

Hence every indefinite integral is absolutely continuous.

4.12. Lemma. If f is absolutely continuous on [a,b], then it is of bounded variation on [a,b].

Proof. Let δ be a positive real number which satisfies the condition in the definition for $\varepsilon = 1$. Select a natural number

$$n > \frac{b-a}{\delta}$$

Consider the partition $\pi = \{x_0, x_1, \ldots, x_n\}$ of [a,b] defined by

$$x_i = x_0 + \frac{i(b-a)}{n}$$

for every i = 0, 1,…, n. Since $|x_i - x_{i-1}| < \delta$, it follows that

$$V_{x_{i-1}}^{x_i} (f) < 1.$$

This implies

$$V_a^b(f) = \sum_{i=1}^{n} V_{x_{i-1}}^{x_i}(f) < n$$

Hence f is of bounded variation.

4.13. Corollary. If f is absolutely continuous, then f has a derivative almost everywhere.

Proof: Since f is absolutely continuous, then by above theorem, f is of bounded variation and hence f has a derivative almost everywhere (by theorem 4.3).

4.14. Lemma. If f is absolutely continuous on [a,b] and f'(x) = 0 a.e., then f is constant.

Proof. We wish to show that f(a) = f(c) for any $c \in [a,b]$.

Let $E \subset (a,c)$ be the set of measure c-a in which f'(x) = 0, and let ε and η be arbitrary positive numbers. To each x in E there is an arbitrarily small interval [x, x+h] contained in [a,c] such that

$|f(x+h) - f(x)| < \eta h$

By Vitali Lemma we can find a finite collection $\{[x_k, y_k]\}$ of non-overlapping intervals of this sort which cover all of E except for a set of measure less than δ, where δ is the positive number corresponding to ε in the definition of the absolute continuity of f . If we label the x_k so that $x_k \le x_{k+1}$, we have (or if we order these intervals so that)

$$a = y_0 \le x_1 < y_1 \le x_2 < \ldots . < y_n \le x_{n+1} = c$$

and

$$f\left(\sum_{k=0}^{n} |x_{k+1} - y_k|\right) < \delta$$

Now, $\sum_{k=0}^{n} |(y_k) - f(x_k)| < \eta \sum_{k=1}^{n} |y_k - x_k| < \eta(c-a)$

by the way to intervals $\{[x_k, y_k]\}$ were constructed, and

$$\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \varepsilon$$

by the absolute continuity of *f*. Thus

$$|f(c) - f(a)| = \sum_{k=0}^{n}[\,|f(x_{k+1}) - f(y_k)|\,] + \sum_{k=1}^{n}[\,|f(y_k) - f(x_k)]\,| \le \varepsilon + \eta(c-a)$$

Since ε and η are arbitrary positive numbers, f(c) − f(a) = 0 and so f(c) = f(a).

Hence f is constant.

4.15. Theorem. A function F is an indefinite integral if and only if it is absolutely continuous.

Proof. Let function F is an indefinite integral then

$F(x) = \int_a^x f(x)dt$

where f is integrable on [a, b]

Now f is integrable on [a, b]

⇨ $|f|$ is integrable on [a, b]. Then for given $\varepsilon > 0$, there is a $\delta > 0$ such that for every measurable set A contained in [a, b] with m(A) < δ, we have $\int_A |f| < \varepsilon$

Let $\{(x_i, x_i')\}^n_{i=1}$ be any finite collection of pairwise disjoint interval in [a, b] such that $\sum_{i=1}^n |x_i - x'_i| < \delta$

Let A = $\bigcup_{i=1}^n (x_i, x'_i)$

Then m(A) = $\sum_{i=1}^n |x_i - x'_i| < \delta$

Therefore we have $\int_A |f| < \varepsilon$

i.e., $\int_{\bigcup_{i=1}^n (x_i, x'_i)} |f| < \varepsilon$

=> $\sum_{i=1}^n \int_{x_i}^{x'_i} |f| < \varepsilon$ …(1)

Consider $\sum_{i=1}^n |F(x'_i) - F(x_i)| = \sum_{i=1}^n \left|\int_a^{x'_i} f(t)dt - \int_a^{x_i} f(t)dt\right|$

$$= \sum_{i=1}^n \left|\int_{x_i}^{x'_i} f(t)dt\right|$$

$$\leq \sum_{i=1}^n \int_{x_i}^{x'_i} |f(t)|\, d(t)$$

$$< \varepsilon [by\ (1)]$$

Conversely, Suppose F is absolutely continuous on [a,b]. Then F is of bounded variation and we may write

$F(x) = F_1(x) - F_2(x)$,

where the functions F_i are monotone increasing. Hence F'(x) exists almost everywhere and $|F'(x)| \leq F_1'(x) + F_2'(x)$

Thus $\int |F'(x)|dx \leq F_1(b) + F_2(b) - F_1(a) - F_2(a)$

and F'(x) is integrable. Let

$$G(x) = \int_a^x F'(t)dt$$

Then G is absolutely continuous and so is the function f = F−G. But by the above lemma since

f'(x) = F'(x)– G'(x)= 0 a.e., we have f to be a constant function. That is,

F(x) − G(x) = A (constant)

or

$F(x) = \int_a^x F'(t)dt = A$

or

$F(x) = \int_a^x F'(t)dt + A$

Taking x = a, we have A = F(a) and so

$F(x) = \int_a^x F'(t)dt + F(a)$

Thus F(x) is indefinite integral of F'(x).